A magnetic model with a possible Chern-Simons phase

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Appendix by F. Goodman† and H. Wenzl‡

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Abstract

A rather elementary family of local Hamiltonians $H_{\circ,\ell}, \ell = 1, 2, 3, \ldots,$ is described for a $2-$dimensional quantum mechanical system of spin $= \frac{1}{2}$ particles. On the torus, the ground state space $G_{\circ,\ell}$ is essentially infinite dimensional but may collapse under "perturbation" to an anyonic system with a complete mathematical description: the quantum double of the $SO(3)-$Chern-Simons modular functor at $q = e^{2\pi i/\ell+2}$ which we call $DE\ell$. The Hamiltonian $H_{\circ,\ell}$ defines a quantum loop gas. We argue that for $\ell = 1$ and 2, $G_{\circ,\ell}$ is unstable and the collapse to $G_{\epsilon,\ell} \cong DE\ell$ can occur truly by perturbation. For $\ell \geq 3$ $G_{\circ,\ell}$ is stable and in this case finding $G_{\epsilon,\ell} \cong DE\ell$ must require either $\epsilon > \epsilon_\ell > 0$, help from finite system size, surface roughening (see section 3), or some other trick, hence the initial use of quotes " ". A hypothetical phase diagram is included in the introduction.

The effect of perturbation is studied algebraically: the ground state $G_{\circ,\ell}$ of $H_{\circ,\ell}$ is described as a surface algebra and our ansatz is that perturbation should respect this structure yielding a perturbed ground state $G_{\epsilon,\ell}$ described by a quotient algebra. By classification, this implies $G_{\epsilon,\ell} \cong DE\ell$. The fundamental point is that nonlinear structures

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may be present on degenerate eigenspaces of an initial $H_\circ$ which constrain the effective action of a perturbation.

There is no reason to expect that a physical implementation of $G_{\epsilon,\ell} \cong DE\ell$ as an anyonic system would require the low temperatures and time asymmetry intrinsic to Fractional Quantum Hall Effect (FQHE) systems $-$ the only currently known physical systems modeled by topological modular functors. A solid state realization of $DE3$, perhaps even one at a room temperature, might be found by building and studying systems, "quantum loop gases" whose main term is $H_{\circ,3}$. This is a challenge for solid state physicists of the present decade. For $\ell = 3, 5, 6, 7, 8, \ldots$, a physical implementation of $DE\ell$ would yield an inherently fault-tolerant universal quantum computer. But a warning must be posted, the theory at $\ell = 2$ is not computationally universal and the first universal theory at $\ell = 3$ seems somewhat harder to locate because of the stability of the corresponding loop gas. Does nature abhor a quantum computer?

Contents

0 Introduction

We write down a positive semidefinite local Hamiltonian $H_{\circ,\ell}$ for a system of locally interacting Ising spins on a $2-$dimensional triangular lattice or surface triangulation, $\ell = 1, 2, 3, \ldots$. In the presence of topology, e.g. on a torus, $H_{\circ,\ell}$ has a highly degenerate space $G_{\circ,\ell}$ of zero modes. We argue for an ansatz which exploits the peculiarly rigid algebraic structure of $G_{\circ,\ell}$ - it is a monoidal tensor category with a unique nontrivial ideal. The ansatz allows us to model any "perturbed" ground state space $G_{\epsilon,\ell}$ (which is itself stable to perturbation) uniquely as a known anyonic system or in mathematical parlance a modular functor. The functor is the Drinfeld double of the even−label−sector of the $SU(2)$-Chern-Simons unitary topological modular functor at level $\ell, DE\ell$. The stability at $\ell = 3$ is probably very slight − see footnote 6 in section 3 and the corresponding discussion.

The Hamiltonian $H_{\circ,\ell}$ defines a quantum loop gas which can be compared with the classical analog. The statistical mechanics of classical loop gases [Ni] identifies a known critical regime and from this we infer that for $\ell = 1$ and 2, $G_{\circ,\ell}$ is unstable and the collapse to $G_{\epsilon,\ell} \cong DE\ell$ is truly by perturbation, for $\ell \geq 3$ $G_{\circ,\ell}$ is stable and in this case $G_{\epsilon,\ell} \cong DE\ell$ requires $\epsilon > \epsilon_\ell > 0$, or some other device (see section 3), hence the initial use of quotes " ". Figure 0.1 is a hypothetical phase diagram.

The reader should not be alarmed that a "doubled" Chern-Simons theory arises. We will explain that the double, being achiral, is more likely to have a robust physical realization. The modular functor $DE\ell$ has $\lambda = \left(\lfloor \frac{\ell+1}{2} \rfloor\right)^2$ "labels" or , physically, λ super selection sectors for quasiparticle excitations (including the empty particle.) Physically this means that a local bit of material, a two dimensional disk, which is in its unique ground state $G_{\epsilon,\ell}$ can have λ types of point-like anyonic excitations (presumably with exponentially decaying tails) occurring in pairs. λ is the number of Gaussian integers $x + iy$ with $0 \leq x, y \leq \ell$, and $x, y =$ even. By mathematically deleting small neighborhoods of such excitations a ground state vector is approximately achieved in the highly degenerate ground state $G_{\epsilon,\ell}$ associated to a punctured disk with "boundary conditions". It is known [FLW2], [FKLW], and [K1] that for $\ell = 3, 5, 6, 7, 8, \ldots$, there is a universal, inherently fault-tolerant, model for quantum computation based on the ability to create, braid, fuse, and finally distinguish these excitations types. Thus $H_{\circ,\ell}$ could be of technological importance: a physical system, a "quantum loop gas", in this (perturbed) universality class could be the substrate of a universal fault tolerant quantum computer.

Any unit vector $\Psi \in G_{\circ,\ell}$ is a superposition of classical $\pm$−spin states $|\Psi\rangle$ which are distinguished by the eigenvalues ± 1 of a commuting family of Pauli matrices σ_z^v equal to $\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$ at vertex v. Sampling $\Psi = \Sigma a_i |\Psi_i\rangle$ by observing $\{\sigma_z^v\}$, we "observe" a classical $|\Psi_i\rangle$

with probability $|a_i|^2$. The domain wall γ_i separating the + - spin regions from $-$ - spin regions of Ψ may be thought of as a random loop gas state [Ni]. It is a Gibbs state with parameters $k = 0$ (self dual) and $n = \left(2\cos\frac{\pi}{\ell+2}\right)^2$, where the weight of a configuration γ is $w(\gamma) = e^{-k(\text{total length }\gamma)} n^{\#\text{ components }\gamma}$. It is know that for $0 < n \leq 2$ and $k = 0$ the loop gas is critical, sitting at a 2^{nd} order phase transition as k crosses from negative to positive. This information together with sections 2 or 3 support a hypothetical phase diagram in parameters $d = 2\cos\frac{\pi}{\ell+2} = \sqrt{n}$ and ϵ. The parameter ϵ scales the perturbation term ϵV which for concreteness may be taken to be $V = \left(\sum_{\text{site } i} \sigma_z^i\right)$.

The challenge to solid state physics is to find or engineer a two dimensional quantum medium in the universality class, $DE3$ below.

The presumptive approach- nearly universal in the literature - to building a quantum computer is quite different from our topological/anyonic starting point. It is based on manipulating and protecting strictly local - as opposed to global or topological - degrees of freedom. It may be called the "qubit approach" since often a union of $2-$level systems (with state space $\bigotimes_i \mathbb{C}_i^2$) is proposed. Actually, the number of levels $-$ or even their finiteness $-$ is not the essential feature, it is that each tensor factor of the state space $-$ call it a qubit $-$ is physically localized in space (or momentum space). The environment will – despite the best efforts of the experimentalist $-$ interacted directly with these "raw" qubits. It has long been recognized ([S], [U]) that the raw qubits must be encrypted into fewer "logical qubits." The demon in this approach is that very low initial (or raw) error rate - perhaps one error per 10^{-5} operations - and large ratios of raw to logical qubits $\sim 10^3$ seem to be required [Pr], to have a stable computational scheme. This problem pervades all approaches based on local or "qubit" systems: liquid NMR, solid state NMR, electron spin, quantum dot, optical cavity, ion trap, etc...

Kitaev's seminal paper [K1] on anyonic computation, amplified in numerous private conversations, provides the foundation for the approach described here. Anyons are a $(2+1)-$dimensional phenomena: when sites containing identical particles in a $2-$dimensional system are exchanged (without collision) there are, up to deformation, two basic exchanges; a clockwise and a counter-clockwise half turn - or "braid" if the motion is considered as generating world lines in $2+1-$dimensional space-time. The two are inverse to each other but

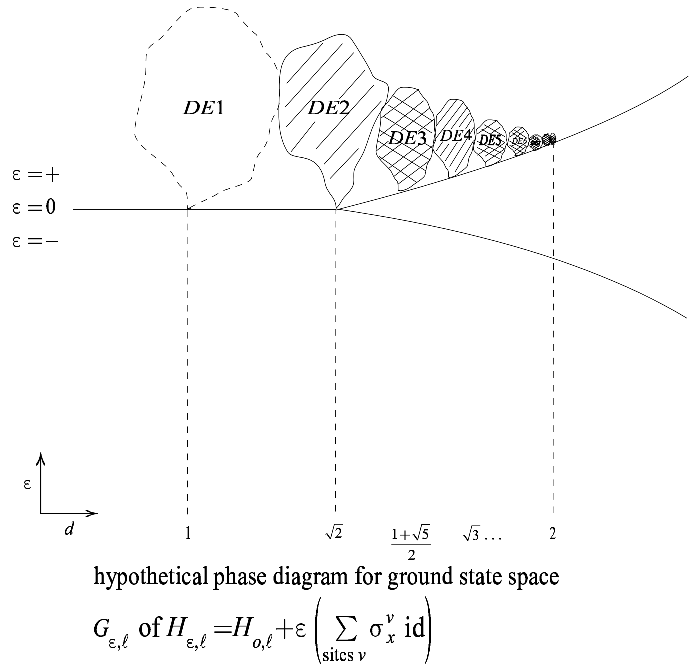

Figure 1: $DE1$ is completely trival: On any surface the ground state is non degenerate and no non-trival particles exists; it is not a genuine topological phase. Shaded regions are the topological phases $DE2, DE3, DE4, \ldots$. Doubly shaded regions are the universal phases $DE3, DE5, DE6, DE7, \ldots$. We have no way of predicting if the topological phases are actually in contact with each other as drawn. Solid lines are phase boundaries between inequivalent systems.

of infinite order rather than order = 2. So whereas only the permutation needs to be recorded for exchanges in R^3, in R^2 "statistics" becomes a representation ρ of a braid group B into the unitary group of a Hilbert space h encoding the internal degrees of freedom of the particle system:

$$\rho : B \longrightarrow U(h).$$

Since a representation into unitary transformations, "gate set" in (quantum) computer science language, is the heart of quantum computation it is not really a surprise that any kind of particle system with a sufficiently general (it certainly must be nonabelian) image $\rho(B)$ can be used to build a universal model for quantum computation.

What are the advantages and disadvantages of anyonic verses qubit computation? The most glaring disadvantage of anyons is that no one is absolutely sure that nonabelian anyons exist in any physical system. Two dimensional electron liquids exhibiting the fractional quantum Hall effect FQHE are the most widely studied candidates for anyonic systems. The Laughlin state at filling fraction $\nu = 1/3$ has observed excitations charges of $(1/3)e$ and these are convincingly linked by the mathematical model with a statistical factor of $\omega = \pm e^{2\pi i/3}$ for the exchange of such pairs. Quasiparticle excitations with nonabelian statistics is one of the most exciting predictions of Chern-Simons theory as a model for the fractional quantum Hall effect FQHE. With a few low level (e.g. $\ell = 1, 2$ or 4, when $G = SU(2)$) exceptions nonabelian anyonic systems are capable, under braiding, of realizing universal quantum computation [FLW2]. The essential point is that the braid representation "Jones representation" associated to the Lie group $SU(2)$ has a dense image at least in $SU(h) \subset U(h)$, h an irreducible summand of the representation. At $\nu = 5/2$ according to [RR] the Hall fluid is modeled by CS4 [the Chern-Simons theory of SU(2) at the 4$^{\text{th}}$ root of unity (level $\ell = 2$)], a theory with a nonabelian "Clifford group" representation. This model was selected from conformal field theories to match expected ground state degeneracies and central charge, and is further supported by numerical evidence on the overlap of trial wave functions. Though very interesting, this representation is still discrete and is not universal in the sense of [FLW1]. However at $\nu = 8/5$ with perhaps weaker numerical support [RR], it is thought that the Hall fluid is modeled by CS5 (level= 3, 5$^{\text{th}}$ root of unity). Here braiding and fusing the excitation would yield universal quantum computation [FLW1]. So let us, for the sake of discussion, accept that FQHE systems have computationally universal anyons. These are very delicate systems:

1. The required crystals have been grown successfully only in a few laboratories.
2. The temperatures at which the finer plateaus are stable are order milK.
3. The chiral asymmetry intrinsic to the effect requires an enormous transverse magnetic field, order 10-15 Tesla. (For CS4 and CS5 the central charge is $\frac{3}{2}$ and $\frac{9}{5}$ respectively.)

Perhaps because of these parameters, ordained by the weakness of the Coulomb interaction at feasible magnet lengths[1], even the most basic experiments to prove existence of "nonabelions" have not been carried out, and use of these systems for computations appears unrealistic.

For applications such as breaking the cryptographic scheme RSA, it can be estimated that several thousand anyons must be formed, braided at will (perhaps implementing ten thousand half twists), and finally fused. This appears to be a nearly impossible task in a FQHE system.

The main point of this paper is that computationally universal anyons may be available in much more convenient systems. Locally $H_{\circ,\ell}$ is a model for a paramagnetic system of Ising spins with short range antiferromagnetic properties. From a more global view point $H_{\circ,\ell}$ describes a quantum loop gas. Written out in products of Pauli matrices $H_{\circ,\ell}$ is seventh order (on the standard triangular lattice) and thus looks complicated compared to, say, the Heisenberg magnet. But geometrically it is quite simple and its ground states are known exactly. A $2-$dimensional material in the universality class $DE\ell$ proposed as the ground state space for $H_{\epsilon,\ell}, \ell \neq 1, 2,$ or 4, will have excitations - "quasiparticles" - capable of universal fault tolerant quantum computation within a model that allows creation, braiding, fusion and measurement of quasiparticle type. We propose searching for such a material.

Perhaps some version of $H_{\circ,\ell} -$ maybe expressed in a different variables $-$ already exists and is waiting to be discovered. Or perhaps with $H_{\circ,\ell}$ in mind something in its (perturbed) universality class can be engineered. If this is possible there would be no reason to expect

[1] In semiconductors with dialectic constant $\epsilon \approx 10$ and $|B| \approx 10$ Tesla the characteristic length $\ell = \sqrt{\epsilon\hbar/e\beta} \approx 150\mathring{A}$.

the system to be particularly delicate. The characteristic energies for magnets are often several hundred Kelvin [NS]. Furthermore the modular functor $DE\ell$ (this includes the information of the various braid group representations and fusion matrices) which arises is amphichiral, the central charge $c = 0$, so there is no reason that time symmetry must be broken and no apparent need for a strong transverse magnetic field. These two features are in marked contrast to FQHE systems.

We make no proposal here for a specific implementation of $H_{\circ,\ell}$ or for how to trap and braid its excitations but we hope that models in the spirit of [NS] for the high T_c cuprates may soon be proposed. In this regard, we note that relatively simple - but still non classical-braiding statics have been proposed [FS] in conjunction with the phenomena of spin-charge separation [A] for high T_c cuprate super conductors above their T_c. Certain -1 phases are predicted to occur when braiding the electron fragments "visions" and "chargeons" around each other and around ground state defects called "holons". Whether this is realized in any known superconductor is open but is cited as precedent for anyonic models for solid state magnetic systems with high characteristic energies. So while the FQHE motivates this paper, we hope we have steered toward its mathematical beauty and away from it experimental difficulties.

What are the generic advantages of anyonic computation? First information is stored in topological properties "large scale entanglement" of the system that cannot be altered (or read) by local interaction. This affords a kind of physical - level stability against error rather than the kind of combinatorial error correction scheme envisioned in the qubit models. Second, at least in the simplest analysis[2], one expects excitation of a stable system to be well localized with exponentially decaying tails. Thus physical braiding should approximate mathematical braiding, $\rho : B \longrightarrow U(h)$ up to a "tunneling" error of the form e^{-cL}, where c is a positive constant, and L is a microscopic length scale describing how well separated the excitations are kept during the braiding process. This error scaling is highly desirable and seems to have no analog in qubit models. While tunneling treats virtual errors, errors which borrow energy briefly from the vacuum, actual errors would be expected scale like $e^{-cT/T_\circ}$ where $T_\circ$ is a character energy for the system and T the operating temperature. This is essentially the error analysis Kitaev made for his anyonic system, the

[2]Kivelson and Sandih[KS] find that Landau level-mixing in FQHE can thicken the tails to polynumerical decay, but this is not a fundamental effect.

tonic code [K1].

This paper draws on three sources of inspiration: 1) Kitaev's paper [K1] on anyonic computation, 2) the FQHE, and 3) rigidity in the classification of von Neumann algebras. Rigidity says that certain monoidal tensor categories have very few ideals. But when interpreted physically, "ideal" means "definable by local conditions", so we find that a certain locality assumption (Ansatz 3.9) strongly limits the physics. This provides an algebraic approach to the perturbation theory of $H_{\circ,\ell}$ - and perhaps yields greater insight than would be possible by analytic methods. We find that for $H_{\circ,\ell}$ the highly degenerate space of $0-$modes $G_{\circ,\ell}$ possess in addition to its linear structure, an important "multiplicative" structure -the structure of a monoidal tensor category - which we argue, should be preserved under a perturbation. The rigidity of type II_1 factor pairs, an aspect of which is stated as Thm 2.1, offers a unique candidate for the (still highly degenerate) "perturbed" ground state space $G_{\epsilon,\ell}$ of $H_{\epsilon,\ell}$ as a representation space. (The induced representation of braid groups is by an adiabatic motion of quasiparticle excitations.)

Throughout, the excitations are assumed to be localized near points so excited states of $H_{\epsilon,\ell}(\Sigma)$ become ground states of $H_{\epsilon,\ell}(\Sigma^-)$ but now on a punctured surface Σ^- with "boundary conditions" or more exactly "labels," (see section 2.) We treat excited states indirectly as ground states on the more complicated surface Σ^-.

We turn now to the definition of $H_{\circ,\ell}$ and return later to amplify on the relations to quantum computing, $\mathbb{C}^*-$ algebras, Chern-Simons theory, and topology. (The connection between Chern-Simons Theory and complexity classes is discussed in [F1].) The existence of a stable phase $G_{\epsilon,\ell} \cong DE\ell$ will be argued by analogy with the FQHE where topological phases are found to be stable, from algebraic uniqueness and via "consistency checks". But these arguments constitute neither a mathematical proof nor a numerical verification. The latter may be exactly as far off as a working quantum computer. It was precisely the problem of studying quantum mechanical Hamiltonians in the thermodynamic limit, e.g. questions of spectral gap, that lead Feynmann [Fe] to dream of the quantum computer in the first place. It is curiously self-referential that we may need a quantum computer to "prove" numerically that a given physical system works like one.

In addition to Alexei Kitaev, I would like to thanks Christian Borgs, Jennifer Chayes, Chetan Nayak, Oded Schramm, Kevin Walker, and Zhenghan Wang for stimulating conversations on the proposed

model.

1 The model

We describe a model system of spin $= \frac{1}{2}$ particles located at the vertices v of a triangulated surface Σ. The Hilbert space is $\mathcal{H} = \bigotimes_{v=1}^{n} \mathbb{C}_v^2$ where $\mathbb{C}_v^2$ is the local degree of freedom $\{|+\rangle, |-\rangle\}$ at the vertex v. The basic Hamiltonian $H_{\circ,\ell}$ is written out below as a sum of local projections. The ground state space (energy $= 0$ vectors) $G_{\circ,\ell}$ of $H_{\circ,\ell}$ can be completely understood (this is unusual since the projector above do not commute) and identified (as $n \longrightarrow \infty$) with what we call the even Temperley-Lieb surface "algebra" ETL_d^s where $d = 2\cos\frac{\pi}{\ell+2}$.

Ultimately our focus will be on the ground states on a multiply punctured disk $-$ the puncture corresponding to anyonic excitations (see section 5). Two issues arise: (1) non-trivial topology and (2) boundary conditions. The boundary conditions are quite tricky so it is best to work first with closed surfaces of arbitrary genus (even though these are not our chief interest) to understand the influence to topology alone "liberated" from boundary conditions.

Y will denote a compact oriented surface throughout. In combinatorial contexts, Y will be given a triangulation $\triangle$ with dual cellulation $\mathcal{C}$. Initially, we consider the case where Y is closed, boundary $Y = \partial Y = \emptyset$. We will speak in terms of the dual cellulation by $2-$cells or "plaquets" c. For example, if Y is a torus it may be cellulated with regular hexagons. This is a perfectly good example to keep in mind but higher genus surfaces are also interesting, while the sphere is less so. Soon we will consider surfaces with boundary.

Distributing $\bigotimes$ over $\underset{\text{on plaquets}}{\Sigma}$, one writes $\mathcal{H} =$ span $\{$classical spin configurations on plaquets$\}$ $=:$ span $\{s_i\}$. Let c be a plaquet, and s classical spin configuration and $\overline{s}$ (or $\overline{s}^c$ for clarity) that configuration with <u>reversed</u> spin ($+ \longrightarrow -$ and $- \longrightarrow +$) at c. For $1 < i, j \leq 2^n$ define $h_{ij}(c) = 1$ if $(1) s_j = \overline{s}_i^c$ and $(2) s_i$ assigns the same spin $\pm$ to c and all its immediate neighbors, and $h_{ij}(c) = 0$ otherwise. Define $g_{ij}(c) = 1$ if $(1) s_j = \overline{s}_i^c$ and (2) the domain wall γ_{s_i} between $+$ and $-$ plaquets, in the spin configuration s_i, meets c in a single connected topological arc, and $g_{ij}(c) = 0$ otherwise. On a surface Y without boundary, we define:

$$\begin{aligned} H_{\circ,\ell} = & \sum_{\substack{\text{plaquets } c, \text{ pairs} \\ \text{of spin states } s_i, s_j}} g_{ij}(c)\,((|s_i\rangle - |s_j\rangle)(\langle s_i| - \langle s_j|)) + \\ & \kappa \sum_{\substack{\text{plaquets } c, \text{ pairs} \\ \text{of spin states } s_i, s_j}} h_{ij}(c)\left((|s_i\rangle - \frac{1}{d}|s_j\rangle)(\langle s_i| - \frac{1}{d}\langle s_j|)\right) \end{aligned} \tag{1}$$

The constant κ is positive and may, in this paper, be set as $\kappa = 1$. To help digest the notation each of the two sums has $n2^{2n}$ terms most of which are zero. It is easy to see that $g_{ij} = g_{ji}$ (If the domain wall γ meets c in a topological arc reversing the spin of c isotopes the domain wall across c to the complementary arc $= \overline{\partial c \smallsetminus \gamma}$.) Contrariwise if $h_{ij} = 1$ then $h_{ji} = 0$. The parameter d could be any positive real number but we will be concerned mainly with $d = 2\cos\frac{\pi}{\ell+2}$, $\ell = 1, 2, 3, \cdots$. The cases $\ell = 2$, $d = \sqrt{2}$ and $\ell = 3$, $d = \frac{1+\sqrt{5}}{2}$, the golden ratio, are of particular interest. Finally, each term in the definition of $H_{\circ,\ell}$ should be read, according to the usual ket-bra notation, as orthogonal projection onto the indicated vector: $|s_i\rangle - |s_j\rangle$ or $|s_i\rangle - 1/d\,|s_j\rangle$. These vectors (whose projectors occur nontrivially in the sums) are certainly not orthogonal to each other (using the inner product $|+\rangle$ hermitian orthonormal to $|-\rangle$ in $\mathbb{C}^2$, extended to define the tensor product Hermitian structure on $\mathcal{H}$) so those individual projectors do not commute. It is therefore surprising at first that we can completely describe the (space of) zero modes $G_{\circ,\ell}$ of this positive semidefinite form, $H_{\circ,\ell}$. However once the description is given the surprise will evaporate for it will be clear how $H_{\circ,\ell}$ was "engineered" precisely to yield this result. Identifying $G_{\circ,\ell}$ is the goal of the remainder of section 1.

Associate to the closed oriented surface Y an infinite dimensional vector space $\mathrm{ETL}_d(Y)$, the even Temperley-Lieb space of Y. It is the formal $\mathbb{C}$ - span of "generalized isotopy classes" of closed bounding $1-$manifolds γ. The "bounding" condition means that γ can be regarded as a $1-$submanifold of Y, a possibly disconnected domain wall, separating Y into a "$|+\rangle$" and a "$|-\rangle$" regions (which may themselves be disconnected.) We do not orient γ, so we do not distinguish here between states which differ by globally interchanging $|+\rangle$ and $|-\rangle$. The term "$1-$manifold" means γ does not branch or terminate at any point. Isotopy, of course, means gradual deformation and generalized isotopy (g - isotopy) extends this equivalence relation by the closing under the rule that if a component $\gamma_\circ$ of γ bonds a disk in Y then declare $\gamma \equiv d(\gamma \setminus \gamma_\circ)$, d times the submanifold with $\gamma_\circ$ deleted. If there is risk of confusion, the notation "$g(d)-$isotopy" will be used.

If Y is compact with nonempty boundary $\partial Y \neq \emptyset$ we will always mark a base point $*$ on each boundary component. We may now similarly define $\mathrm{ETL}_d(Y)$ to be the $\mathbb{C}-$span of $g(d)-$isotopy classes of properly imbedded bounding $1-$manifolds γ, $\partial\gamma \subset \partial Y$, with γ and the isotopy γ_t, $0 \leq t \leq 1$, disjoint from all base points. We extend the definition of $g-$isotopy to permit the isotopy of endpoints of γ within a boundary component C minus its base point, $C \setminus *$. Classical configurations containing a <u>bigon</u> are "killed" by multiplying the coefficient by zero. In the bulk $g-$isotopy is as before. A "bigon" is by definition a maximal region of constant spin $+$ or $-$ in Y which is topologically a disk and meets ∂Y is a single connected topological arc.

Given a triangulation $\triangle$ of Y we may restrict to those γ which arise as the boundary of a union of dual $2-$cells or plaquets of $\mathcal{C}$.

Definition 1.1. Suppose γ_1 and γ_2 are both closed, bounding, dual $1-$manifolds (as above) of a closed surface Y and each has a complex weight c_i. Then we say (γ_1, c_1) and (γ_2, c_2) are $\triangle - g-$isotopic if there is a path from γ_1 to γ_2 consisting of combination of two moves (1) deforming the current $1-$manifold γ across some $\triangle-$dual $2-$cell (plaquet) that it meets in a topological arc and (2) adding or removing a circle of γ bounding a plaquet (and adjusting the numerical weight by a factor of d or $1/d$ respectively).

Note 1.2. If the plaquets determining γ are those of constant spin $+$ or $-$, then moves in lemma 1.1 correspond to the two types of terms in the definition of $H_{\circ,\ell}$.

Now suppose γ_i is the domain wall between $+$ and $-$ plaquets on a surface Y with boundary.

Definition 1.3. If γ_1 and γ_2 are domain walls for two spin assigments on a surface Y which agree on ∂Y, then we say they are $\triangle - g-$isotopic if they are related by a sequence of moves which include (1) and (2) above in the bulk and in addition ($1'$) deforming the current γ across plaquet c meeting ∂Y provided c does not contain a base point and γ meets c in a connected topological arc, and ($2'$) if a plaquet c meeting ∂Y is itself a bigon, then the weight associated to the configuration γ (which contains $\partial c \setminus \partial Y$) is multiplied by zero.

Beginning with a triangulation $\triangle$ on a surface Y, with or without boundary, we may construct a combinatorial model $\mathrm{ETL}_d^{\triangle}(Y) = \mathbb{C}$ - span {bounding dual 1−manifolds $\gamma/\triangle - g$−isotopy}. There is a natural map of $\mathbb{C}$−vector spaces:

$$\mathrm{ETL}_d^{\triangle}(Y) \to \mathrm{ETL}_d(Y). \tag{2}$$

These maps are of course never onto (only the simpler g−isotopy classes are realized) and Walker and Wang have observed that for certain triangulations $\triangle$ the kernel can also be non-zero. However, it is easy to see that as $\triangle$ is subdivided. $\mathrm{ETL}_d^{\triangle}(Y)$ approximates $\mathrm{ETL}_d(Y)$ in the sense that the direct limit $\varinjlim \mathrm{ETL}_d^{\triangle}(Y) \cong \mathrm{ETL}_d(Y)$.

With $\triangle$ a fixed triangulation of our closed surface Y, set $G_{\circ,\ell}(Y,\triangle) =$ ground state space (zero modes) of the positive semidefinate $H_{\circ,\ell}$ defined above (1.1). Let $^-$ denote the global involution exchanging $|+\rangle$ and $|-\rangle$. Clearly $H_{\circ,\ell}$ is $^-$−invariant and so $G_{\circ,\ell}$ is $^-$−invariant. Note that $^-$ is not always fixed point free (e.g. on the 2 - sphere, $\Sigma = S^2$, (all $|+\rangle$) $\equiv d(|+\rangle$ in North , $|-\rangle$ in south) $\equiv$ (all $|-\rangle$) $\equiv$ (all $|+\rangle)^-$. Also on $Y = T^2$, if s is a classical state which is $|+\rangle$ on an essential annulus $A \subset T^2$ and $|-\rangle$ on $T^2 \setminus A$ then $s \equiv \bar{s}$. In both cases $\equiv$ means g−isotopy of the domain wall. Let $G^+_{\circ,\ell}(Y,\triangle)$ denote the $+1$−eigenspace of $^-$.

Proposition 1.4. *For Y a closed surface, there is a natural isomorphism $G^+_{\circ,\ell}(Y,\triangle) \cong \left(\mathrm{ETL}_d^{\triangle}(Y)\right)^*$.*

Proof: Let $\mathrm{OETL}_d^{\triangle}(Y) =$ span (2−colorings with domain all $\gamma)/g$−isotopy be the oriented, even, Temperley-Lieb space. Given a subspace of a Hilbert space $R \subset \mathcal{H}$, $R^\perp$ consists of functionals on $\mathcal{H}$ which carry the "relation subspace" R to zero. So it is immediate from (2) that $\Psi \in G_\circ(Y,\triangle) \subset \mathcal{H}$ iff $\Psi = \sum\limits_i (\zeta|s_i\rangle)|s_i\rangle$ for some linear functional $\zeta : \mathrm{OETL}_d^{\triangle} \longrightarrow \mathbb{C}$. Thus $G_{\circ,\ell}(Y,\triangle) \cong \left(\mathrm{OETL}_d^{\triangle}(Y)\right)^*$. The involution $^-$ acts compatibility on both sides and $(\mathrm{ETL}_d^{\triangle}(Y))^*$ may be identified as the $+1$ eigenspace of $^-$ on the r.h.s. $\square$

To discuss the boundary "edge" or $\partial(Y)$ at a physical level, it is necessary to introduce $H^\partial_{\circ,\ell}$, the basic Hamiltonian at level ℓ on a surface (Y,∂) with $\partial \neq \emptyset$. In the bulk, $H^\partial_{\circ,\ell} = H_{\circ,\ell}$. On ∂Y the first ("isotopy") term of $H_{\circ,\ell}$ is interpreted as allowing the ends of domain walls to fluctuate on ∂Y provided that the fluctuation does not carry a domain wall across a base point $* \subset C \subset \partial Y$. The second ("circle") term of $H_{\circ,\ell}$ is modified to create an energy penalty to the existence of a boundary bigon c; i.e. a placquet meeting a boundary component C and having opposite spin from all its neighbors.

$$\begin{aligned} H^{\partial}_{\circ,\ell} = & \sum_{\substack{\text{plaquets } c, \text{ meeting} \\ C, \text{ states } s_i, s_j}} g^{\partial}_{ij}(c)\,((|s_i\rangle - |s_j\rangle)(\langle s_i| - \langle s_j|)) + \\ & \kappa \sum_{\substack{\text{plaquets } c, \text{ meeting} \\ C, \text{ states } s_k}} h^{\partial}_k(c)(|s_k\rangle\langle s_k|), + H_{\circ,\ell} \text{ in bulk} \end{aligned} \tag{3}$$

Define $g^{\partial}_{ij}(c) = 1$ if (1) c meets C but is disjoint from its base point $*$, (2) $s_j = \overline{s}^c_i$ and (3) the domain wall γ_{s_i} meets c in a single connected arc; and $g^{\partial}_{ij}(c) = 0$ otherwise. Define $h^{\partial}_k(c) = 1$ if c meets C and c has spin opposite to each of its neighbors and $h^{\partial}_k(c) = 0$ otherwise.

Just as in the closed case, it is easy to give a topological description of the ground state space $G^{\partial}_{\circ,\ell}(Y, \partial Y)$ of $H^{\partial}_{\circ,\ell}$ on a surface with boundary. The ground states $G^{\partial}_{\circ,\ell}$ are the $\mathbb{C}-$span of equivalence classes of $\triangle-$vertex $2-$colorings with no topological bigons. The equivalence relation is $g(d)-$isotopy in the bulk and isotopy on the boundary. The absence of bigons results from the second term of $H^{\partial}_{\circ,\ell}$ $-$ which kills single plaquet bigons $-$ when coupled with the fluctuations created by the first term.

The insertion of a base point $*$ on each boundary component $C \subset \partial Y$ and the corresponding adjustment of line (3) is directly analogous to the framing of Wilson loop in [Wi], in fact the base point moving in time defines the first direction of a normal frame to the Wilson loop in the $2+1$ dimensional space-time picture. As in the previous application, the base point is introduced for mathematical rather than physical reasons. It allows the state vectors in each conformal block to be identified precisely and not merely up to a (block-dependent) phase ambiguity. Concretely in our model the base point prevents domain walls from spinning around a puncture. Note that if (a superposition of) domain walls γ represent an eigenspace for Dehn twist around the picture with eigenvalue $\lambda \neq 1$ and if twisting is not prevented then the relation $|\gamma\rangle = \lambda|\gamma\rangle$ will occur, killing the state $|\gamma\rangle$ which is certainly not desired. I thank Nayak for pointing out that although choosing base points breaks symmetry, none of the physics depends on which base points are chosen. Thus Hamiltonian in the bounded case has a $\underbrace{U(1) \times \cdots \times U(1)}_{k}-$gauge